The Gabriel's Horn

Gabriel's Trumpet (also called Torricelli's Trumpet) primarily refers to a famous geometric mathematical figure, conceived by Evangelista Torricelli in the 17th century.

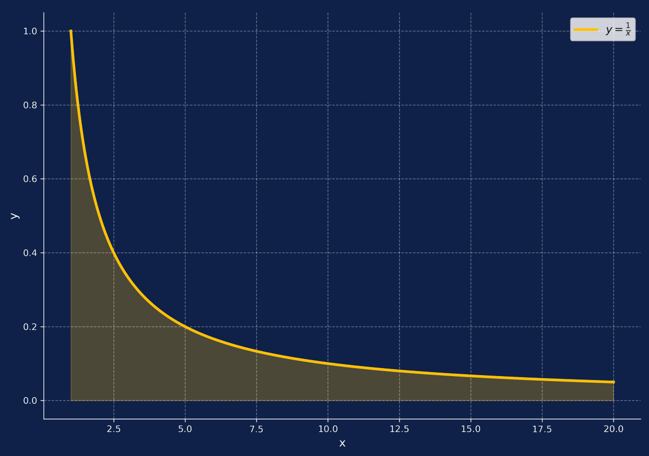
It is known for a striking paradox:

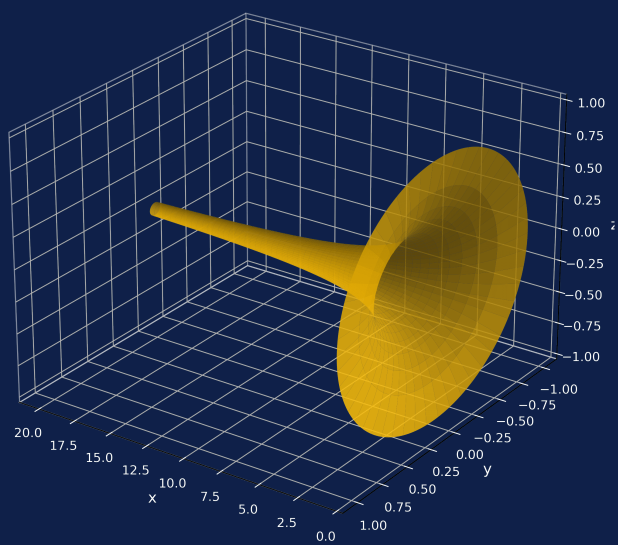
- Its volume is finite (it could theoretically be filled with a finite amount of paint).
- Its surface area is infinite (its interior could never be entirely painted).

Does it seem impossible ? Yet, it's mathematically true !

What is Gabriel’s Horn?

To construct this object, we consider the curve with equation $y = \frac{1}{x}$ defined on the interval $x \in [1, +\infty[$. We then rotate this curve around the x-axis (the x axis). We will calculate the volume, then the surface area.





Volume

The formula to calculate the volume of a solid of revolution around the x-axis is given by:

$$V = \int_a^b \pi [f(x)]^2 dx$$

with a and b the bounds of the interval on which the function is defined.

The integral over the volume V from 1 to infinity is:

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2} dx$$
$$= \pi \left[-\frac{1}{x}\right]_1^\infty$$
$$= \pi \left(\lim_{t \to \infty} \left(-\frac{1}{t}\right) - \left(-\frac{1}{1}\right)\right)$$
$$= \pi(0 - (-1))$$
$$V = \pi$$

Conclusion : The volume is equal to π. Therefore, it is finite.

Surface area

The formula for the lateral surface area A of a solid of revolution generated by $f(x)$ is given by:

$$A = \int_1^\infty 2\pi f(x)\sqrt{1+(f'(x))^2}dx$$

$$A = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$$

The area of the trumpet is greater than the integral of $\frac{1}{x}$:

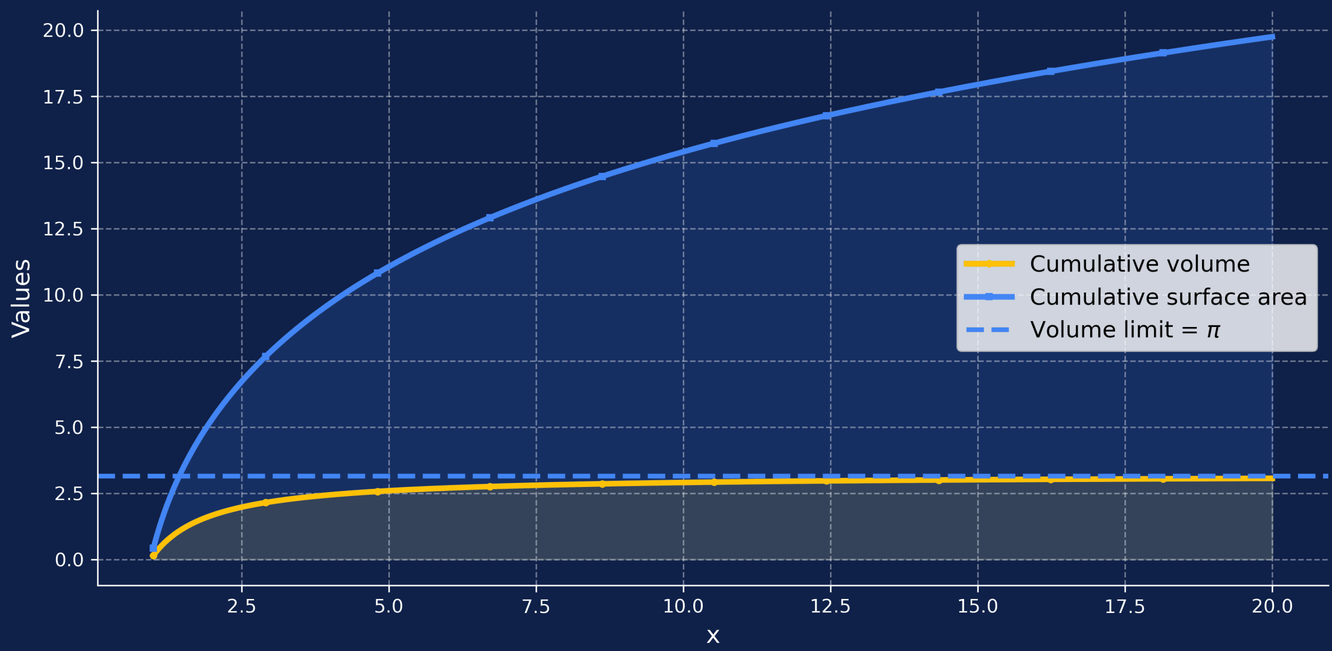
$$A > 2\pi \int_1^\infty \frac{1}{x}dx$$

Now, we know that the integral of $\frac{1}{x}$ (the logarithm function) diverges to infinity:

$$\int_1^\infty \frac{1}{x}dx = [\ln(x)]_1^\infty = \infty - 0 = \infty$$

Conclusion : Since A is greater than a value that tends towards infinity, the area A is infinite.

Visualizing



The mathematical result leads to an absurd physical situation:

- The inside of the trumpet can be filled with a pot of paint of volume π.
- However, this quantity of paint would theoretically not be enough to cover the inner surface if one tried to paint it with a brush.

About MathSpark today's contributor

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"As a Mathematical Modeling and Numerical Simulation Engineer, I am passionate about using mathematics and Artificial Intelligence to solve real-world problems. My goal is to contribute significantly to innovative initiatives and to develop advanced mathematical and computational solutions."

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